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A NOVEL INTEGRAL-TYPE CLR CONDITION FOR COMMON FIXED POINTS OF WEAKLY COMPATIBLE MAPPINGS IN NEUTROSOPHIC METRIC SPACES WITH APPLICATIONS TO DYNAMIC PROGRAMMING

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Abstract. This paper presents a new common fixed point theorem for two pairs of weakly compatible mappings in neutrosophic metric spaces under an integral-type contractive condition formulated through the common limit range (CLR) property. The proposed theorem generalizes, improves, and unifies several well-known results in fixed point theory. To highlight its practical significance, the main result is applied to establish the existence and uniqueness of solutions for functional equations encountered in multistage decision processes within the framework of dynamic programming. A nontrivial example is included to illustrate the effectiveness of the theoretical results.

1. Introduction

Advancements in fixed point theorems across various abstract spaces have broad theoretical significance. These theorems play a crucial role in numerous mathematical disciplines, including systems of ordinary and partial differential equations, game theory, variational inequalities, and functional equations that arise in dynamic programming. Atanassov [2] introduced the concept of Intuitionistic Fuzzy Sets (IFS), which generalizes fuzzy sets and effectively captures uncertain, incomplete, or vague information, especially when hesitation is involved. Building on IFS, Park [3] developed intuitionistic fuzzy metric spaces using continuous t-norms and t-conorms, extending the notion of fuzzy metric spaces. Subsequently, Alaca et al. [4] extended Banach's fixed point theorems to this framework. Further, Turkoglu et al. [5] established common fixed point theorems within intuitionistic fuzzy metric spaces. In 1984, Khan et al. [8] enhanced Banach's contraction principle by introducing altering distance functions as control functions, which have become important tools in fixed point theory. Wairojjana et al. [9] demonstrated fixed point results in complete and compact fuzzy metric spaces utilizing altering distance functions. Sintunavarat and Kumam [10] introduced the CLR_T property, eliminating the need for closedness of the range space. Later, Chauhan et al. [13] proposed the CLR_{ST} property, which plays a pivotal role in the developments discussed in this paper.

In 2002, Branciari [14] introduced an integral version of the Banach contraction principle. Following this, Murthy et al. [15] developed common fixed point theorems for self-maps within fuzzy metric spaces. Later, Alsulami et al. [16] extended integral-type contractive mappings by incorporating α-admissible functions in metric spaces. In 2014, Liu established fixed point theorems for weakly compatible mappings satisfying integral contraction conditions in metric spaces, providing illustrative examples. Sarwar et al. [18] in 2015 contributed fixed point results alongside uniqueness criteria for solutions to certain functional equations. More recently, Gupta et al. [19] obtained common fixed point results for pairs of w-compatible mappings in fuzzy metric spaces. In 2020, Shatanawi et al.

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[20] explored common fixed point theorems in modified intuitionistic generalized fuzzy metric spaces by utilizing the E.A. property.

In this study, we combine the concepts of altering distance functions, the CLR_{ST} property, and integral-type contractive mappings to establish fixed point theorems within the framework of neutrosophic metric spaces. Our primary objective is to investigate the existence of solutions to functional equations arising in dynamic programming by applying common fixed point results for two pairs of weakly compatible mappings that satisfy a novel integral-type condition in the setting of neutrosophic metric spaces.

We recollect some definitions and theorems which are usable for establishment of our main results.

Definition 1.1 (t-norm [1]). *A binary operation $\lhd : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if $\lhd$ fulfils the following conditions:*
(i) $\lhd$ is commutative and associative ;
(ii) $\lhd$ is continuous ;
(iii) $u \lhd 1 = u, \forall u \in [0,1]$;
(iv) $u \lhd v \leq s \lhd t$, whenever $u \leq s$ and $v \leq t$, for all $u, v, s, t \in [0,1]$.

Definition 1.2 (t-conorm [1]). *A binary operation $\oplus : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-conorm if $\oplus$ shares the following conditions:*
(i) $\oplus$ is commutative and associative;
(ii) $\oplus$ is continuous;
(iii) $u \oplus 0 = u, \forall a \in [0,1]$;
(iv) $u \oplus v \leq s \oplus t$, whenever $u \leq s$ and $v \leq t$, for all $u, v, s, t \in [0,1]$.

Definition 1.3. [1] *A 5-tuple $(X, H, K, *, \oplus)$ is called an intuitionistic fuzzy metric space if $*$ is a continuous t-norm, $\oplus$ is a continuous t-conorm, and H, K are fuzzy sets defined on $X^2 \times [0, \infty)$ fulfil the following properties, for every $x, y, z \in X, t > 0$.*
(i) $H(x,y,t) + K(x,y,t) \leq 1$;
(ii) $H(x,y,t) > 0, H(x,y,0) = 0$;
(iii) $H(x,y,t) = 1$ iff $x = y$;
(iv) $H(x,y,t) \lhd H(y,z,s) \leq H(x,z,t+s)$;
(v) $H(x,y,\cdot) : [0,\infty) \to [0,1]$ is continuous;
(vi) $H(x,y,t) = H(y,x,t)$;
(vii) $K(x,y,t) > 0, K(x,y,0) = 1$;
(viii) $K(x,y,t) = 0$ iff $x = y$;
(ix) $K(x,y,t) \oplus K(y,z,s) \geq K(x,z,t+s)$;
(x) $K(x,y,\cdot) : [0,\infty) \to [0,1]$ is continuous;
(xi) $K(x,y,t) = K(y,x,t)$;
(xii) $\lim H(x,y,t) = 1$ when $t \to \infty, \forall x, y \in X$;
(xiii) $\lim K(x,y,t) = 0$ when $t \to \infty, \forall x, y \in X$.
The pair (H, K) is called an intuitionistic fuzzy metric space on X.

Definition 1.4. *Let $(X, H, *)$ be a fuzzy metric space, and let P, Q, R, S be self-mappings on X. The pairs (P, Q) and (R, S) are said to possess the joint common limit in the range (JCLR) property if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that*

$$\lim_{n\to\infty} \mathcal{P}x_n = \lim_{n\to\infty} \mathcal{Q}x_n = \lim_{n\to\infty} \mathcal{R}y_n = \lim_{n\to\infty} \mathcal{S}y_n = \mathcal{Q}v = \mathcal{S}v$$

for some $v \in X$.

Turkoglu, Alaca and Yildiz [6] introduced the notions of compatible mappings in intuitionistic fuzzy metric space as follows:

Definition 1.5. [7] *Two self-maps A and B on the space $(X, H, K, *, \oplus)$ are called weakly compatible if they commute whenever they share a common fixed point.*

Definition 1.6. [10] *Consider an intuitionistic fuzzy metric space $(X, H, K, *, \oplus)$ along with four self-maps f, g, h, and j. The pairs (f, j) and (g, h) are said to fulfill the common limit range property relative to j and h, denoted by CLR_{jh}, if there exist two sequences $\{u_n\}$ and $\{v_n\}$ in X such that*

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} ju_n = \lim_{n\to\infty} gv_n = \lim_{n\to\infty} hv_n = \ell \in j(X) \cap h(X).$$

Mihet et al. [11] proved fixed point results using altering distance functions within the framework of complete probabilistic Menger spaces.

Definition 1.7. [12] *A function $\kappa : [0, 1] \to [0, 1]$ is called an altering distance function if it meets the following conditions:*
(i) κ is non-decreasing;
(ii) κ is continuous;
(iii) $\kappa(\ell) = 0$ holds true if and only if $\ell = 0$.

2. Main results

Definition 2.1. [22] *A 6 tuple $(X, \mathscr{G}, \mathscr{K}, \mathscr{P}, *, \oplus)$ is said to be neutrosophic metric space if $\lhd$ is a continuous t-norm, $\oplus$ is a continuous t-conorm and $\mathscr{G}, \mathscr{D}, \mathscr{P}$ are fuzzy sets on $X^2 \times [0, \infty]$ which fulfil the following properties, for every $x, y, z \in X, t > 0$.*

(1) $\mathscr{G}(x, y, t) + \mathscr{D}(x, y, t) + \mathscr{P}(x, y, t) \leq 3$;
(2) $\mathscr{G}(x, y, t) > 0$, $\mathscr{G}(x, y, 0) = 0$;
(3) $\mathscr{G}(x, y, t) = 1$ *if and only if* $x = y$;
(4) $\mathscr{G}(x, y, t) \lhd \mathscr{G}(y, z, s) \leq \mathscr{G}(x, z, t + s)$;
(5) $\mathscr{G}(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*
(6) $\mathscr{G}(x, y, t) = \mathscr{G}(y, x, t)$;
(7) $\mathscr{D}(x, y, t) \geq 0$, $\mathscr{D}(x, y, 0) = 1$;
(8) $\mathscr{D}(x, y, t) = 0$ *if and only if* $x = y$;
(9) $\mathscr{D}(x, y, t) \rhd \mathscr{D}(y, z, s) \geq \mathscr{D}(x, z, t + s)$;
(10) $\mathscr{D}(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*
(11) $\mathscr{D}(x, y, t) = \mathscr{D}(y, x, t)$;
(12) $\mathscr{P}(x, y, t) \geq 0$, $\mathscr{P}(x, y, 0) = 1$;
(13) $\mathscr{P}(x, y, t) = 0$ *if and only if* $x = y$;
(14) $\mathscr{P}(x, y, t) \rhd \mathscr{P}(y, z, s) \geq \mathscr{P}(x, z, t + s)$;
(15) $\mathscr{P}(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is continuous;*
(16) $\mathscr{P}(x, y, t) = \mathscr{P}(y, x, t)$;
(17) $\lim_{t\to\infty} \mathscr{G}(x, y, t) = 1$, *for all* $x, y \in X$;
(18) $\lim_{t\to\infty} \mathscr{D}(x, y, t) = 0$, *for all* $x, y \in X$;
(19) $\lim_{t\to\infty} \mathscr{P}(x, y, t) = 0$, *for all* $x, y \in X$.

The triple $(\mathscr{G}, \mathscr{D}, \mathscr{P})$ is called a neutrosophic metric *on X.*

Definition 2.2. [21] *Let $(X, \mathscr{G}, \mathscr{D}, \mathscr{P}, \lhd, \rhd)$ be a neutrosophic metric space and let f, g, h, and j be four self-mappings on X. The pairs (f, j) and (g, h) are said to satisfy the* common limit range property *with respect to j and h, denoted by CLR_{jh}, if there exist sequences $\{u_n\}$ and $\{v_n\}$ in X such that:*

$$\lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} j(u_n) = \lim_{n\to\infty} g(v_n) = \lim_{n\to\infty} h(v_n) = \ell \in j(X) \cap h(X),$$

where the limit is taken with respect to the neutrosophic metric, i.e., convergence in the sense that:

$$\lim_{n\to\infty} \mathscr{G}(f(u_n), \ell, t) = 1, \quad \lim_{n\to\infty} \mathscr{D}(f(u_n), \ell, t) = 0, \quad \lim_{n\to\infty} \mathscr{P}(f(u_n), \ell, t) = 0$$

and similarly for the other terms.

Theorem 2.1. *Consider the neutrosophic metric space $(X, \mathscr{G}, \mathscr{D}, \mathscr{P}, \lhd, \rhd)$, where the t-norm and t-conorm are defined by $a \lhd b = \min\{a, b\}$ and $a \oplus b = \max\{a, b\}$, respectively. Let f, g, h, and j be self-maps on X. For all $x, y \in X$ and $t > 0$, we have*

$$\int_0^{\mathscr{G}(fx,gy,s)} \hat{\mu}(s)ds \geq \mho \left(\int_0^{\bar{\lambda}(x,y,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{D}(fx,gy,s)} \hat{\mu}(s)ds \leq \phi \left(\int_0^{\hat{\nu}(x,y,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{P}(fx,gy,s)} \hat{\mu}(s)ds \leq \Phi \left(\int_0^{W(x,y,s)} \hat{\mu}(s)ds \right)$$

where $\hat{\mu} : \mathbb{R}^+ \to \mathbb{R}$ is a Lebesgue integrable function that is non-negative and summable, $\phi, |Phi = \{\phi, \Phi \mid \phi, \Phi : \mathbb{R} \to \mathbb{R}$ is upper semi-continuous, $\phi(0) = 0, \phi(s) < s, \Phi(0) = 0$, and $\Phi(s) < s\}$ for all $s > 0$, and $\mho = \{\mho \mid \mho : \mathbb{R} \to \mathbb{R}$ is a comparison function, lower semi-continuous, with $\mho(s) > s$ for all $s > 0\}$,
$\bar{\lambda}(x, y, s) = \mathscr{G}(fx, gy, s) \lhd \mathscr{G}(fx, hx, s) \lhd \mathscr{G}(hx, jy, s) \lhd \mathscr{G}(gy, jy, s) \lhd \mathscr{G}(hx, gy, s) \lhd \mathscr{G}(fx, jy, s)$ and $\hat{\nu}(x, y, s) = \mathscr{D}(fx, gy, s) \oplus \mathscr{D}(fx, hx, s) \oplus \mathscr{D}(hx, jy, s) \oplus \mathscr{D}(gy, jy, s) \oplus \mathscr{D}(hx, gy, s) \oplus \mathscr{D}(fx, jy, s)$ and $W(x, y, s) = \mathscr{P}(fx, gy, s) \oplus \mathscr{P}(fx, hx, s) \oplus \mathscr{P}(hx, jy, s) \oplus \mathscr{P}(gy, jy, s) \oplus \mathscr{P}(hx, gy, s) \oplus \mathscr{P}(fx, jy, s)$. Additionally, assume that the pairs (f, h) and (g, j) possess the common limit range (CLR) property. Moreover, if both pairs (f, h) and (g, j) are weakly compatible, then the mappings f, g, h, and j share a unique common fixed point in X.

Proof. Assume that the pairs (f, h) and (g, j) fulfill the common limit range property. Consequently, there exist sequences $\{u_n\}$ and $\{v_n\}$ such that

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} hu_n = \lim_{n\to\infty} gv_n = \lim_{n\to\infty} jv_n = w. \tag{1}$$

for a certain $w \in j(X) \cap h(X)$. Because w lies in $h(X)$, there exists a point $z \in X$ with $h(z) = w$. Using equation (1), it follows that

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} hu_n = \lim_{n\to\infty} gv_n = \lim_{n\to\infty} jv_n = w = hz. \tag{2}$$

Next, we assert that $f(z) = h(z)$. Assume, for the sake of contradiction, that $f(z) \neq h(z)$.
By replacing x with z and y with v_n in the hypothesis of Theorem 2.1, we derive

$$\int_0^{\mathscr{G}(fz,gv_n,s)} \hat{\mu}(s)ds \geq \mho \left(\int_0^{\bar{\lambda}(z,v_n,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{D}(fz,gv_n,s)} \hat{\mu}(s)ds \leq \phi \left(\int_0^{\hat{\nu}(z,v_n,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{P}(fz,gv_n,s)} \hat{\mu}(s)ds \leq \Phi \left(\int_0^{W(z,v_n,s)} \hat{\mu}(s)ds \right).$$

$$\bar{\lambda}(z, v_n, s) = \mathscr{G}(fz, gv_n, s) \lhd \mathscr{G}(fz, hz, s) \lhd \mathscr{G}(hz, jv_n, s) \lhd \mathscr{G}(gv_n, jv_n, s) \lhd \mathscr{G}(hz, gv_n, s) \lhd \mathscr{G}(fz, jv_n, s),$$
$$\hat{\nu}(z, v_n, s) = \mathscr{D}(fz, gv_n, s) \oplus \mathscr{D}(fz, hz, s) \oplus \mathscr{D}(hz, jv_n, s) \oplus \mathscr{D}(gv_n, jv_n, s) \oplus \mathscr{D}(hz, gv_n, s) \oplus \mathscr{D}(fz, jv_n, s)$$
$$W(z, v_n, s) = \mathscr{P}(fz, gv_n, s) \oplus \mathscr{P}(fz, hz, s) \oplus \mathscr{P}(hz, jv_n, s) \oplus \mathscr{P}(gv_n, jv_n, s) \oplus \mathscr{P}(hz, gv_n, s) \oplus \mathscr{P}(fz, jv_n, s).$$

Taking the limit as $n \to \infty$ in the above inequalities yields

$$\begin{aligned}
\lim_{n\to\infty} \bar{\lambda}(z, v_n, s) &= \mathscr{G}(fz, w, s) \lhd \mathscr{G}(fz, w, s) \lhd \mathscr{G}(w, w, s) \lhd \mathscr{G}(w, w, s), \lhd \mathscr{G}(w, w, s) \lhd \mathscr{G}(fz, w, s) \\
&= \mathscr{G}(fz, w, s) \lhd \mathscr{G}(fz, w, s) \lhd 1 \lhd 1 \lhd 1 \lhd \mathscr{G}(fz, w, s) \\
&= \mathscr{G}(fz, w, s) \lhd \mathscr{G}(fz, w, s) \lhd \mathscr{G}(fz, w, s) \\
&= \min\{\mathscr{G}(fz, w, s), \mathscr{G}(fz, w, s), \mathscr{G}(fz, w, s)\} \\
&= \mathscr{G}(fz, w, s) \text{ and} \\
\lim_{n\to\infty} \hat{\nu}(z, v_n, s) &= \mathscr{D}(fz, w, s) \oplus \mathscr{D}(fz, w, s) \oplus \mathscr{D}(w, w, s) \oplus \mathscr{D}(w, w, s) \oplus \mathscr{D}(w, w, s) \oplus \mathscr{D}(fz, w, s) \\
&= \mathscr{D}(fz, w, s) \oplus \mathscr{D}(fz, w, s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathscr{D}(fz, w, s) \\
&= \mathscr{D}(fz, w, s) \oplus \mathscr{D}(fz, w, s) \oplus \mathscr{D}(fz, w, s) \\
&= \max\{\mathscr{D}(fz, w, s) \oplus \mathscr{D}(fz, w, s) \oplus \mathscr{D}(fz, w, s)\} \\
&= \mathscr{D}(fz, w, s) \\
\lim_{n\to\infty} W(z, v_n, s) &= \mathscr{P}(fz, w, s) \oplus \mathscr{P}(fz, w, s) \oplus \mathscr{P}(w, w, s) \oplus \mathscr{P}(w, w, s) \oplus \mathscr{P}(w, w, s) \oplus \mathscr{P}(fz, w, s) \\
&= \mathscr{P}(fz, w, s) \oplus \mathscr{P}(fz, w, s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathscr{P}(fz, w, s) \\
&= \mathscr{P}(fz, w, s) \oplus \mathscr{P}(fz, w, s) \oplus \mathscr{P}(fz, w, s) \\
&= \max\{\mathscr{P}(fz, w, s) \oplus \mathscr{P}(fz, w, s) \oplus \mathscr{P}(fz, w, s)\} \\
&= \mathscr{P}(fz, w, s)
\end{aligned}$$

Furthermore

$$\begin{aligned}
\int_0^{\mathscr{G}(fz,w,s)} \hat{\mu}(s)ds &= \lim_{n\to\infty} \inf \int_0^{\mathscr{G}(fz,gv_n,s)} \hat{\mu}(s)ds \\
&\geq \lim_{n\to\infty} \inf \mho \left(\int_0^{\bar{\lambda}(z,v_n,s)} \hat{\mu}(s)ds \right) \\
&\geq \mho \left(\lim_{n\to\infty} \inf \int_0^{\bar{\lambda}(z,v_n,s)} \hat{\mu}(s)ds \right) \\
&= \mho \left(\int_0^{\mathscr{G}(fz,w,s)} \hat{\mu}(s)ds \right) \\
&> \int_0^{\mathscr{G}(fz,w,s)} \hat{\mu}(s)ds
\end{aligned}$$

and

$$\begin{aligned}\int_0^{\mathscr{D}(fz,w,s)} \hat{\mu}(s)ds &= \lim_{n\to\infty}\sup \int_0^{\mathscr{D}(fz,gy_n,s)} \hat{\mu}(s)ds \\ &\leq \lim_{n\to\infty}\sup \phi\left(\int_0^{\hat{\nu}(z,y_n,s)} \hat{\mu}(s)ds\right) \\ &\leq \phi\left(\lim_{n\to\infty}\sup \int_0^{\hat{\nu}(z,y_n,s)} \hat{\mu}(s)ds\right) \\ &= \phi\left(\int_0^{\mathscr{D}(fz,w,s)} \hat{\mu}(s)ds\right) \\ &< \int_0^{\mathscr{D}(fz,w,s)} \hat{\mu}(s)ds\end{aligned}$$

and

$$\begin{aligned}\int_0^{\mathscr{P}(fz,w,s)} \hat{\mu}(s)ds &= \lim_{n\to\infty}\sup \int_0^{\mathscr{P}(fz,gy_n,s)} \hat{\mu}(s)ds \\ &\leq \lim_{n\to\infty}\sup \Phi\left(\int_0^{W(z,y_n,s)} \hat{\mu}(s)ds\right) \\ &\leq \Phi\left(\lim_{n\to\infty}\sup \int_0^{W(z,y_n,s)} \hat{\mu}(s)ds\right) \\ &= \Phi\left(\int_0^{\mathscr{P}(fz,w,s)} \hat{\mu}(s)ds\right) \\ &< \int_0^{\mathscr{P}(fz,w,s)} \hat{\mu}(s)ds\end{aligned}$$

which leads to a contradiction.
Thus, we conclude that $f(z) = h(z)$. Therefore,

$$fz = hz = w. \tag{3}$$

Because w belongs to $j(X)$, there exists some $\rho \in X$ for which $j(\rho) = w$. Replacing $j(\rho) = w$ into equation (1), we get

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} hu_n = \lim_{n\to\infty} gv_n = \lim_{n\to\infty} jv_n = w = j\rho. \tag{4}$$

Next, we aim to show that $g(\rho) = j(\rho)$. Assume, on the contrary, that $g(\rho) \neq j(\rho)$.
By substituting $x = u_n$ and $y = \rho$ into the hypothesis of Theorem 2.1, we derive

$$\int_0^{\mathscr{G}(fu_n,g\rho,s)} \hat{\mu}(s)ds \geq \mho\left(\int_0^{\bar{\lambda}(u_n,\rho,s)} \hat{\mu}(s)ds\right)$$

and

$$\int_0^{\mathscr{D}(fu_n,g\rho,s)} \hat{\mu}(s)ds \leq \phi\left(\int_0^{\hat{\nu}(u_n,\rho,s)} \hat{\mu}(s)ds\right)$$

and

$$\int_0^{\mathscr{P}(fu_n,g\rho,s)} \hat{\mu}(s)ds \leq \Phi\left(\int_0^{W(u_n,\rho,s)} \hat{\mu}(s)ds\right)$$

$$
\begin{aligned}
\bar{\lambda}\left(u_n,\rho,s\right) =& \mathscr{G}\left(fu_n,g\rho,s\right) \lhd \mathscr{G}\left(fu_n,hu_n,s\right) \lhd \mathscr{G}\left(hu_n,j\rho,s\right) \\
& * \mathscr{G}(g\rho,j\rho,s) \lhd \mathscr{G}\left(hu_n,g\rho,s\right) \lhd \mathscr{G}\left(fu_n,j\rho,s\right) \\
\hat{\nu}\left(u_n,\rho,s\right) =& \mathscr{D}\left(fu_n,g\rho,s\right) \oplus \mathscr{D}\left(fu_n,hu_n,s\right) \oplus \mathscr{D}\left(hu_n,j\rho,s\right) \\
& \oplus \mathscr{D}(g\rho,j\rho,s) \oplus \mathscr{D}\left(hu_n,g\rho,s\right) \oplus \mathscr{P}\left(fu_n,j\rho,s\right) \\
W\left(u_n,\rho,s\right) =& \mathscr{P}\left(fu_n,g\rho,s\right) \oplus \mathscr{P}\left(fu_n,hu_n,s\right) \oplus \mathscr{P}\left(hu_n,j\rho,s\right) \\
& \oplus \mathscr{P}(g\rho,j\rho,s) \oplus \mathscr{P}\left(hu_n,g\rho,s\right) \oplus \mathscr{P}\left(fu_n,j\rho,s\right)
\end{aligned}
$$

Taking the limit as $n \to \infty$ in the inequalities above, we have

$$
\begin{aligned}
\lim_{n\to\infty} \bar{\lambda}\left(u_n,\rho,s\right) &= \mathscr{G}(w,w,s) \lhd \mathscr{G}(w,w,s) \lhd \mathscr{G}(w,w,s) \lhd \mathscr{G}(w,w,s) \lhd \mathscr{G}(w,w,s) \lhd \mathscr{G}(w,w,s) \\
&= \mathscr{G}(w,w,s) \lhd \mathscr{G}(w,w,s) \lhd 1 \lhd 1 \lhd 1 \lhd \mathscr{G}(w,w,s) \\
&= \mathscr{G}(w,w,s) \lhd \mathscr{G}(w,w,s) \lhd \mathscr{G}(w,w,s) \\
&= \min\{\mathscr{G}(w,w,s), \mathscr{G}(w,w,s), \mathscr{G}(w,w,s)\} \\
&= \mathscr{G}(w,w,s) \text{ and} \\
\lim_{n\to\infty} \hat{\nu}\left(u_n,\rho,s\right) &= \mathscr{D}(w,w,s) \oplus \mathscr{D}(w,w,s) \oplus \mathscr{D}(w,w,s), \oplus \mathscr{D}(w,w,s), \oplus \mathscr{D}(w,w,s), \oplus \mathscr{D}(fw,w,s) \\
&= \mathscr{D}(w,w,s) \oplus \mathscr{D}(w,w,s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathscr{D}(w,w,s) \\
&= \mathscr{D}(w,w,s) \oplus \mathscr{D}(w,w,s) \oplus \mathscr{D}(w,w,s) \\
&= \max\{\mathscr{D}(w,w,s) \oplus \mathscr{D}(w,w,s) \oplus \mathscr{D}(w,w,s)\} \\
&= \mathscr{D}(w,w,s) \text{ and} \\
\lim_{n\to\infty} W\left(u_n,\rho,s\right) &= \mathscr{P}(w,w,s) \oplus \mathscr{P}(w,w,s) \oplus \mathscr{P}(w,w,s), \oplus \mathscr{P}(w,w,s), \oplus \mathscr{P}(w,w,s), \oplus \mathscr{P}(fw,w,s) \\
&= \mathscr{P}(w,w,s) \oplus \mathscr{P}(w,w,s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathscr{P}(w,w,s) \\
&= \mathscr{P}(w,w,s) \oplus \mathscr{P}(w,w,s) \oplus \mathscr{P}(w,w,s) \\
&= \max\{\mathscr{P}(w,w,s) \oplus \mathscr{P}(w,w,s) \oplus \mathscr{P}(w,w,s)\} \\
&= \mathscr{P}(w,w,s)
\end{aligned}
$$

In addition

$$
\begin{aligned}
\int_0^{\mathscr{G}(w,w,s)} \hat{\mu}(s)ds &= \lim_{n\to\infty} \inf \int_0^{\mathscr{G}(fu_n,g\rho,s)} \hat{\mu}(s)ds \\
&\geq \lim_{n\to\infty} \inf \mho\left(\int_0^{\bar{\lambda}(u_n,\rho,s)} \hat{\mu}(s)ds\right) \\
&\geq \mho\left(\lim_{n\to\infty} \inf \int_0^{\bar{\lambda}(u_n,\rho,s)} \hat{\mu}(s)ds\right) \\
&= \mho\left(\int_0^{\mathscr{G}(w,w,s)} \hat{\mu}(s)ds\right) \\
&> \int_0^{\mathscr{G}(w,w,s)} \hat{\mu}(s)ds
\end{aligned}
$$

and

$$\begin{aligned}\int_0^{\mathscr{D}(w,w,s)} \hat{\mu}(s)ds &= \lim_{n\to\infty}\sup \int_0^{\mathscr{D}(fu_n,g\rho,s)} \hat{\mu}(s)ds \\ &\leq \lim_{n\to\infty}\sup \phi\left(\int_0^{\bar{\lambda}(u_n,\rho,s)} \hat{\mu}(s)ds\right) \\ &\leq \phi\left(\lim_{n\to\infty}\sup \int_0^{\bar{\lambda}(u_n,\rho,s)} \hat{\mu}(s)ds\right) \\ &= \phi\left(\int_0^{\mathscr{D}(w,w,s)} \hat{\mu}(s)ds\right) \\ &< \int_0^{\mathscr{D}(w,w,s)} \hat{\mu}(s)ds\end{aligned}$$

and

$$\begin{aligned}\int_0^{\mathscr{P}(w,w,s)} \hat{\mu}(s)ds &= \lim_{n\to\infty}\sup \int_0^{\mathscr{P}(fu_n,g\rho,s)} \hat{\mu}(s)ds \\ &\leq \lim_{n\to\infty}\sup \Phi\left(\int_0^{\bar{\lambda}(u_n,\rho,s)} \hat{\mu}(s)ds\right) \\ &\leq \Phi\left(\lim_{n\to\infty}\sup \int_0^{\bar{\lambda}(u_n,\rho,s)} \hat{\mu}(s)ds\right) \\ &= \Phi\left(\int_0^{\mathscr{P}(w,w,s)} \hat{\mu}(s)ds\right) \\ &< \int_0^{\mathscr{P}(w,w,s)} \hat{\mu}(s)ds\end{aligned}$$

which results in a contradiction.
Thus, it follows that $g(\rho) = j(\rho)$. Consequently,

$$g\rho = j\rho = w \tag{5}$$

By (3) and (5), we get

$$fz = hz = g\rho = j\rho = w. \tag{6}$$

Next, we demonstrate that w is a common fixed point of the mappings f, g, h, and j.
Given that the pairs (f, h) and (g, j) are weakly compatible,

$$fz = hz \Rightarrow hfz = fhz.$$

Now, applying (6), we get

$$fw = hw \tag{7}$$

Also,

$$g\rho = j\rho \Rightarrow jg\rho = gj\rho$$

From (6), we get

$$gw = jw. \tag{8}$$

Next, we assert that $f(w) = w$. Suppose, for the sake of contradiction, that $f(w) \neq w$.
By substituting $x = w$ and $y = \rho$ into the conditions of Theorem 2.1, we get

$$\int_0^{\mathscr{G}(fw,g\rho,s)} \hat{\mu}(s)ds \geq \mho\left(\int_0^{\bar{\lambda}(w,\rho,s)} \hat{\mu}(s)ds\right)$$

and

$$\int_0^{\mathscr{D}(fw,g\rho,s)} \hat{\mu}(s)ds \leq \phi\left(\int_0^{\hat{\nu}(w,\rho,s)} \hat{\mu}(s)ds\right)$$

and

$$\int_0^{\mathscr{P}(fw,g\rho,s)} \hat{\mu}(s)ds \leq \Phi\left(\int_0^{W(w,\rho,s)} \hat{\mu}(s)ds\right)$$

$\bar{\lambda}(w,\rho,s) = \mathscr{G}(fw,g\rho,s) \lhd \mathscr{G}(fw,hw,s) \lhd \mathscr{G}(hw,j\rho,s), \lhd\mathscr{G}(g\rho,j\rho,s), \lhd\mathscr{G}(hw,g\rho,s), \lhd\mathscr{G}(fw,j\rho,s),$
$\hat{\nu}(w,\rho,s) = \mathscr{D}(fw,g\rho,s) \oplus \mathscr{D}(fw,hw,s) \oplus \mathscr{D}(hw,j\rho,s), \oplus\mathscr{D}(g\rho,j\rho,s), \oplus\mathscr{D}(hw,g\rho,s), \oplus\mathscr{D}(fw,j\rho,s),$
$W(w,\rho,s) = \mathscr{P}(fw,g\rho,s) \oplus \mathscr{P}(fw,hw,s) \oplus \mathscr{P}(hw,j\rho,s), \oplus\mathscr{P}(g\rho,j\rho,s), \oplus\mathscr{P}(hw,g\rho,s), \oplus\mathscr{P}(fw,j\rho,s).$
Now,

$$\begin{aligned}
\lim_{n\to\infty} \bar{\lambda}(w,\rho,s) &= \mathscr{G}(fw,w,s) \lhd \mathscr{G}(fw,w,s) \lhd \mathscr{G}(w,w,s), \lhd\mathscr{G}(w,w,s), \lhd\mathscr{G}(w,w,s), \lhd\mathscr{G}(fw,w,s) \\
&= \mathscr{G}(fw,w,s) \lhd \mathscr{G}(fw,w,s) \lhd 1 \lhd 1 \lhd 1 \lhd \mathscr{G}(fw,w,s) \\
&= \mathscr{G}(fw,w,s) \lhd \mathscr{G}(fw,w,s) \lhd \mathscr{G}(fw,w,s) \\
&= \min\{\mathscr{G}(fw,w,s), \mathscr{G}(fw,w,s), \mathscr{G}(fw,w,s)\} \\
&= \mathscr{G}(fw,w,s) \text{ and} \\
\lim_{n\to\infty} \hat{\nu}(w,\rho,s) &= \mathscr{D}(fw,w,s) \oplus \mathscr{D}(fw,w,s) \oplus \mathscr{D}(w,w,s), \oplus\mathscr{D}(w,w,s), \oplus\mathscr{D}(w,w,s), \oplus\mathscr{D}(fw,w,s) \\
&= \mathscr{D}(fw,w,s) \oplus \mathscr{D}(fw,w,s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathscr{D}(fw,w,s) \\
&= \mathscr{D}(fw,w,s) \oplus \mathscr{D}(fw,w,s) \oplus \mathscr{D}(fw,w,s) \\
&= \max\{\mathscr{D}(fw,w,s) \oplus \mathscr{D}(fw,w,s) \oplus \mathscr{D}(fw,w,s)\} \\
&= \mathscr{D}(fw,w,s) \text{ and} \\
\lim_{n\to\infty} W(w,\rho,s) &= \mathscr{P}(fw,w,s) \oplus \mathscr{P}(fw,w,s) \oplus \mathscr{P}(w,w,s), \oplus\mathscr{P}(w,w,s), \oplus\mathscr{P}(w,w,s), \oplus\mathscr{P}(fw,w,s) \\
&= \mathscr{P}(fw,w,s) \oplus \mathscr{P}(fw,w,s) \oplus 0 \oplus 0 \oplus 0 \oplus \mathscr{P}(fw,w,s) \\
&= \mathscr{P}(fw,w,s) \oplus \mathscr{P}(fw,w,s) \oplus \mathscr{P}(fw,w,s) \\
&= \max\{\mathscr{P}(fw,w,s) \oplus \mathscr{P}(fw,w,s) \oplus \mathscr{P}(fw,w,s)\} \\
&= \mathscr{P}(fw,w,s)
\end{aligned}$$

Additionally

$$\begin{aligned}
\int_0^{\mathscr{G}(fw,w,s)} \hat{\mu}(s)ds &= \lim_{n\to\infty} \inf \int_0^{\mathscr{G}(fw,g\rho,s)} \hat{\mu}(s)ds \\
&\geq \lim_{n\to\infty} \inf \mho\left(\int_0^{\bar{\lambda}(w,\rho,s)} \hat{\mu}(s)ds\right) \\
&\geq \mho\left(\lim_{n\to\infty} \inf \int_0^{\bar{\lambda}(w,\rho,s)} \hat{\mu}(s)ds\right) \\
&= \mho\left(\int_0^{\mathscr{G}(fw,w,s)} \hat{\mu}(s)ds\right) \\
&> \int_0^{\mathscr{G}(fw,w,s)} \hat{\mu}(s)ds
\end{aligned}$$

and

$$\begin{aligned}\int_0^{\mathscr{D}(fw,w,s)} \hat{\mu}(s)ds &= \lim_{n\to\infty}\sup \int_0^{\mathscr{D}(fw,g\rho,s)} \hat{\mu}(s)ds \\ &\leq \lim_{n\to\infty}\sup \phi\left(\int_0^{\bar{\lambda}(w,\rho,s)} \hat{\mu}(s)ds\right) \\ &\leq \phi\left(\lim_{n\to\infty}\sup \int_0^{\bar{\lambda}(w,\rho,s)} \hat{\mu}(s)ds\right) \\ &= \phi\left(\int_0^{\mathscr{D}(fw,w,s)} \hat{\mu}(s)ds\right) \\ &< \int_0^{\mathscr{D}(fw,w,s)} \hat{\mu}(s)ds\end{aligned}$$

and

$$\begin{aligned}\int_0^{\mathscr{P}(fw,w,s)} \hat{\mu}(s)ds &= \lim_{n\to\infty}\sup \int_0^{\mathscr{P}(fw,g\rho,s)} \hat{\mu}(s)ds \\ &\leq \lim_{n\to\infty}\sup \Phi\left(\int_0^{\bar{\lambda}(w,\rho,s)} \hat{\mu}(s)ds\right) \\ &\leq \Phi\left(\lim_{n\to\infty}\sup \int_0^{\bar{\lambda}(w,\rho,s)} \hat{\mu}(s)ds\right) \\ &= \Phi\left(\int_0^{\mathscr{P}(fw,w,s)} \hat{\mu}(s)ds\right) \\ &< \int_0^{\mathscr{P}(fw,w,s)} \hat{\mu}(s)ds\end{aligned}$$

This leads to a contradiction. Hence, we conclude that $f(w) = w$. Using equation (7), we deduce

$$fw = hw = w. \tag{9}$$

Similarly, by replacing $x = z$ and $y = w$ in the hypothesis of Theorem 2.1, and applying equations (6) and (8), we derive

$$gw = jw = w. \tag{10}$$

Combining (9) and(20), we get

$$fw = hw = w = gw = jw.$$

Therefore, w serves as a common fixed point for the mappings f, g, h, and j. Next, we examine the uniqueness of this common fixed point. Suppose w_1 and w_2 are both common fixed points of f, g, h, and j. By substituting $x = w_1$ and $y = w_2$ into the conditions of Theorem 2.1, we obtain

$$\int_0^{\mathscr{G}(w_1,w_2,s)} \hat{\mu}(s)ds = \int_0^{\mathscr{G}(fw_1,gw_2,s)} \hat{\mu}(s)ds \geq \mho\left(\int_0^{\bar{\lambda}(w_1,w_2,s)} \hat{\mu}(s)ds\right)$$

and

$$\int_0^{\mathscr{D}(w_1,w_2,s)} \hat{\mu}(s)ds = \int_0^{\mathscr{D}(fw_1,gw_2,s)} \hat{\mu}(s)ds \leq \phi\left(\int_0^{\hat{\nu}(w_1,w_2,s)} \hat{\mu}(s)ds\right)$$

and

$$\int_0^{\mathscr{P}(w_1,w_2,s)} \hat{\mu}(s)ds = \int_0^{\mathscr{P}(fw_1,gw_2,s)} \hat{\mu}(s)ds \leq \Phi\left(\int_0^{W(w_1,w_2,s)} \hat{\mu}(s)ds\right)$$

$$\begin{aligned}\bar{\lambda}(w_1,w_2,s) =& \mathscr{G}(fw_1,gw_2,s) \lhd \mathscr{G}(fw_1,hw_1,s) \lhd \mathscr{G}(hw_1,jw_2,s) \lhd \\ & \mathscr{G}(gw_2,jw_2,s) \lhd \mathscr{G}(hw_1,gw_2,s) \lhd \mathscr{G}(fw_1,jw_2,s) \\ =& \mathscr{G}(w_1,w_2,s) \lhd \mathscr{G}(w_1,w_1,s) \lhd \mathscr{G}(w_1,w_2,s) \lhd \\ & \mathscr{G}(w_2,w_2,s) \lhd \mathscr{G}(w_1,w_2,s) \lhd \mathscr{G}(w_1,w_2,s) \\ =& \mathscr{G}(w_1,w_2,s) \lhd 1 \lhd \mathscr{G}(w_1,w_2,s) \lhd 1 \lhd \mathscr{G}(w_1,w_2,s) \lhd \mathscr{G}(w_1,w_2,s) \\ =& \mathscr{G}(w_1,w_2,s) \lhd \mathscr{G}(w_1,w_2,s) \lhd \mathscr{G}(w_1,w_2,s) \lhd \mathscr{G}(w_1,w_2,s) \\ =& \min\{\mathscr{G}(w_1,w_2,s), \mathscr{G}(w_1,w_2,s), \mathscr{G}(w_1,w_2,s), \mathscr{G}(w_1,w_2,s)\} \\ =& \mathscr{G}(w_1,w_2,s)\end{aligned}$$

and

$$\begin{aligned}\hat{\nu}(w_1,w_2,s) =& \mathscr{D}(fw_1,gw_2,s) \oplus \mathscr{D}(fw_1,hw_1,s) \oplus \mathscr{D}(hw_1,jw_2,s), \oplus \\ & \mathscr{D}(gw_2,jw_2,s), \oplus \mathscr{D}(hw_1,gw_2,s), \oplus \mathscr{D}(fw_1,jw_2,s) \\ =& \mathscr{D}(w_1,w_2,s) \oplus \mathscr{D}(w_1,w_1,s) \oplus \mathscr{D}(w_1,w_2,s) \oplus \\ & \mathscr{D}(w_2,w_2,s) \oplus \mathscr{D}(w_1,w_2,s) \oplus \mathscr{D}(w_1,w_2,s) \\ =& \mathscr{D}(w_1,w_2,s) \oplus 0 \oplus \mathscr{D}(w_1,w_2,s) \oplus 0 \oplus \mathscr{D}(w_1,w_2,s) \oplus \mathscr{D}(w_1,w_2,s) \\ =& \mathscr{D}(w_1,w_2,s) \oplus \mathscr{D}(w_1,w_2,s) \oplus \mathscr{D}(w_1,w_2,s) \oplus \mathscr{D}(w_1,w_2,s) \\ =& \max\{\mathscr{D}(w_1,w_2,s), \mathscr{D}(w_1,w_2,s), \mathscr{D}(w_1,w_2,s), \mathscr{D}(w_1,w_2,s)\} \\ =& \mathscr{D}(w_1,w_2,s).\end{aligned}$$

and

$$\begin{aligned}W(w_1,w_2,s) =& \mathscr{P}(fw_1,gw_2,s) \oplus \mathscr{P}(fw_1,hw_1,s) \oplus \mathscr{P}(hw_1,jw_2,s), \oplus \\ & \mathscr{P}(gw_2,jw_2,s), \oplus \mathscr{P}(hw_1,gw_2,s), \oplus \mathscr{P}(fw_1,jw_2,s) \\ =& \mathscr{P}(w_1,w_2,s) \oplus \mathscr{P}(w_1,w_1,s) \oplus \mathscr{P}(w_1,w_2,s) \oplus \\ & \mathscr{P}(w_2,w_2,s) \oplus \mathscr{P}(w_1,w_2,s) \oplus \mathscr{P}(w_1,w_2,s) \\ =& \mathscr{P}(w_1,w_2,s) \oplus 0 \oplus \mathscr{P}(w_1,w_2,s) \oplus 0 \oplus \mathscr{P}(w_1,w_2,s) \oplus \mathscr{P}(w_1,w_2,s) \\ =& \mathscr{P}(w_1,w_2,s) \oplus \mathscr{P}(w_1,w_2,s) \oplus \mathscr{P}(w_1,w_2,s) \oplus \mathscr{P}(w_1,w_2,s) \\ =& \max\{\mathscr{P}(w_1,w_2,s), \mathscr{P}(w_1,w_2,s), \mathscr{P}(w_1,w_2,s), \mathscr{P}(w_1,w_2,s)\} \\ =& \mathscr{P}(w_1,w_2,s).\end{aligned}$$

Thus,

$$\begin{aligned}\int_0^{\mathscr{G}(w_1,w_2,s)} \hat{\mu}(s)ds &\geq \mho\left(\int_0^{\mathscr{G}(w_1,w_2,s)} \hat{\mu}(s)ds\right) \\ &> \int_0^{\mathscr{G}(w_1,w_2,s)} \hat{\mu}(s)ds\end{aligned}$$

and

$$\begin{aligned}\int_0^{\mathscr{D}(w_1,w_2,s)} \hat{\mu}(s)ds &\leq \phi\left(\int_0^{\mathscr{D}(w_1,w_2,s)} \hat{\mu}(s)ds\right) \\ &< \int_0^{\mathscr{D}(w_1,w_2,s)} \hat{\mu}(s)ds\end{aligned}$$

and

$$\begin{aligned}\int_0^{\mathscr{P}(w_1,w_2,s)} \hat{\mu}(s)ds &\leq \Phi\left(\int_0^{\mathscr{P}(w_1,w_2,s)} \hat{\mu}(s)ds\right) \\ &< \int_0^{\mathscr{P}(w_1,w_2,s)} \hat{\mu}(s)ds\end{aligned}$$

which leads to a contradiction.
Therefore, we must have $w_1 = w_2$. Consequently, the mappings f, g, h, and j share a unique common fixed point in X. This concludes the proof. □

Example 2.1. *Consider the set $X = [0, 4]$ equipped with a neutrosophic metric $(M, N, \mathscr{P})$ defined by, for all $x, y \in X$ and $t > 0$,*

$$\mathscr{G}(x, y, t) = e^{-\frac{|x-y|}{t+1}}, \quad \mathscr{D}(x, y, t) = 1 - e^{-\frac{|x-y|}{t+1}}, \quad \mathscr{P}(x, y, t) = \frac{|x - y|}{2(t + 1)}.$$

Let the continuous t-norm $\lhd$ be the product $a \lhd b = a \cdot b$, and the t-conorm $\rhd$ be $a \rhd b = a + b - ab$.

Define four self-maps $f, g, h, j : X \to X$ as

$$f(x) = \begin{cases} 2, & x \leq 2 \\ \frac{1}{3}, & x > 2 \end{cases}, \quad g(x) = \begin{cases} 2, & x \leq 2 \\ \frac{1}{4}, & x > 2 \end{cases},$$

$$h(x) = \begin{cases} 2, & x \leq 2 \\ \frac{1}{5}, & x > 2 \end{cases}, \quad j(x) = \begin{cases} 2, & x \leq 2 \\ \frac{1}{3}, & x > 2 \end{cases}.$$

Consider sequences $\{u_n\} = \frac{2}{n}$ and $\{v_n\} = \frac{2}{n+1}$ in X. Then,

$$\lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} g(v_n) = \lim_{n\to\infty} h(u_n) = \lim_{n\to\infty} j(v_n) = 2.$$

Let $\hat{\mu}(t) = 3t$, $\mho(t) = 2t$, and $\phi(t) = \frac{t}{3}$.

For any $x, y \in [0, 2]$, observe

$$f(x) = g(y) = h(x) = j(y) = 2,$$

and thus

$$\mathscr{G}(fx, gy, t) = \mathscr{G}(fx, hx, t) = \mathscr{G}(hx, jy, t) = \mathscr{G}(gy, jy, t) = e^0 = 1,$$

$$\mathscr{D}(fx, gy, t) = \mathscr{D}(fx, hx, t) = \mathscr{D}(hx, jy, t) = \mathscr{D}(gy, jy, t) = 0,$$

and

$$\mathscr{P}(fx, gy, t) = \mathscr{P}(fx, hx, t) = \mathscr{P}(hx, jy, t) = \mathscr{P}(gy, jy, t) = 0.$$

Setting $\bar{\lambda}(x, y, t) = 1$ and $\hat{\nu}(x, y, t) = 0$, and noting

$$\mho\left(\int_0^1 \hat{\mu}(s)ds\right) = \mho(1.5) = 3 > \int_0^1 \hat{\mu}(s)ds = 1.5,$$

the conditions

$$\int_0^{\mathscr{G}(fx,gy,t)} \hat{\mu}(s)ds \geq \mho\left(\int_0^{\bar{\lambda}(x,y,t)} \hat{\mu}(s)ds\right)$$

and

$$\int_0^{\mathscr{D}(fx,gy,t)} \hat{\mu}(s)ds \leq \phi\left(\int_0^{\hat{\nu}(x,y,t)} \hat{\mu}(s)ds\right)$$

and

$$\int_0^{\mathscr{P}(fx,gy,t)} \hat{\mu}(s)ds \leq \Phi\left(\int_0^{W(x,y,t)} \hat{\mu}(s)ds\right)$$

are satisfied.

Hence, by the corresponding fixed-point theorem in neutrosophic metric spaces 2.1, the mappings f, g, h, j share a unique common fixed point in X, which is $x = 2$.

Corollary 2.1. *Consider an neutrosophic metric space $(X, \mathscr{G}, \mathscr{D}, \mathscr{P}, \lhd, \rhd)$, where the t-norm and t-conorm are defined by $a \lhd b = \min\{a, b\}$ and $a \oplus b = \max\{a, b\}$, respectively. Let f, h, and j be self-maps on X. For all $x, y \in X$ and $t > 0$, we have*

$$\int_0^{\mathscr{G}(fx,fy,s)} \hat{\mu}(s)ds \geq \mho \left(\int_0^{\bar{\lambda}(x,y,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{D}(fx,fy,s)} \hat{\mu}(s)ds \leq \phi \left(\int_0^{\hat{\nu}(x,y,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{P}(fx,fy,s)} \hat{\mu}(s)ds \leq \Phi \left(\int_0^{W(x,y,s)} \hat{\mu}(s)ds \right)$$

where $\hat{\mu} : \mathbb{R}^+ \to \mathbb{R}$ is a Lebesgue integrable function that is both summable and non-negative, $\mho = \{\mho \mid \mho : \mathbb{R} \to \mathbb{R}$ is upper semi-continuous, $\mho(0) = 0$, and $\mho(s) < s$ for all $s > 0\}$,
and $\phi, \Phi = \{\phi, \Phi \mid \phi, \Phi : \mathbb{R} \to \mathbb{R}\}$ are comparison functions that are left-continuous, non-decreasing, and satisfy $\phi(s), \Phi(s), \Phi(s) > s, \phi(s) > s$ for all $s > 0$,
$\bar{\lambda}(x, y, s) = \mathscr{G}(fx, fy, s) \lhd \mathscr{G}(fx, hx, s) \lhd \mathscr{G}(hx, jy, s), \lhd \mathscr{G}(fy, jy, s), \lhd \mathscr{G}(hx, fy, s), \lhd \mathscr{G}(fx, jy, s)$ *and*
$\hat{\nu}(x, y, s) = \mathscr{D}(fx, fy, s) \oplus \mathscr{D}(fx, hx, s) \oplus \mathscr{D}(hx, jy, s), \oplus \mathscr{D}(fy, jy, s), \oplus \mathscr{D}(hx, fy, s), \oplus \mathscr{D}(fx, jy, s)$
and $W(x, y, s) = \mathscr{P}(fx, fy, s) \oplus \mathscr{P}(fx, hx, s) \oplus \mathscr{P}(hx, jy, s), \oplus \mathscr{P}(fy, jy, s), \oplus \mathscr{P}(hx, fy, s), \oplus \mathscr{P}(fx, jy, s)$.
Additionally, suppose the pairs (f, h) and (f, j) satisfy the common limit range (CLR) property relative to the mappings h and j, respectively. Moreover, if both pairs (f, h) and (f, j) are weakly compatible, then the mappings f, h, and j share a unique common fixed point in X.

Corollary 2.2. *Let $(X, \mathscr{G}, \mathscr{D}, \mathscr{P}, \lhd, \rhd)$ be a neutrosophic metric space, where the t-norm and t-conorm are given by $a \lhd b = \min\{a, b\}$ and $a \oplus b = \max\{a, b\}$, respectively. Consider two self-maps f and j defined on X, and assume that for all $x, y \in X$ and $t > 0$,*

$$\int_0^{\mathscr{G}(fx,fy,s)} \hat{\mu}(s)ds \geq \mho \left(\int_0^{\bar{\lambda}(x,y,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{D}(fx,fy,s)} \hat{\mu}(s)ds \leq \phi \left(\int_0^{\hat{\nu}(x,y,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{P}(fx,fy,s)} \hat{\mu}(s)ds \leq \Phi \left(\int_0^{W(x,y,s)} \hat{\mu}(s)ds \right)$$

where $\hat{\mu} : \mathbb{R}^+ \to \mathbb{R}$ is a non-negative, summable function that is Lebesgue integrable. Let $\mho = \{\mho \mid \mho : \mathbb{R} \to \mathbb{R}$ is upper semi-continuous with $\mho(0) = 0$ and $\mho(s) < s$ for all $s > 0\}$,
and $\phi, \Phi = \{\phi, \Phi \mid \phi, \Phi : \mathbb{R} \to \mathbb{R}\}$ are comparison functions that are left-continuous, non-decreasing, and satisfy $\phi(s), \Phi(s), \phi(s) > s, \Phi(s) > s$ for every $s > 0$.

$\bar{\lambda}(x, y, s) = \mathscr{G}(fx, fy, s) \lhd \mathscr{G}(fx, jx, s) \lhd \mathscr{G}(jx, jy, s), \lhd \mathscr{G}(fy, jy, s), \lhd \mathscr{G}(jx, fy, s), \lhd \mathscr{G}(fx, jy, s)$
and $\hat{\nu}(x, y, s) = \mathscr{D}(fx, fy, s) \oplus \mathscr{D}(fx, jx, s) \oplus \mathscr{D}(jx, jy, s), \oplus \mathscr{D}(fy, jy, s), \oplus \mathscr{D}(jx, fy, s), \oplus \mathscr{D}(fx, jy, s)$
and $W(x, y, s) = \mathscr{P}(fx, fy, s) \oplus \mathscr{P}(fx, jx, s) \oplus \mathscr{P}(jx, jy, s), \oplus \mathscr{P}(fy, jy, s), \oplus \mathscr{P}(jx, fy, s), \oplus \mathscr{P}(fx, jy, s)$.
Assume further that the pair (f, j) satisfies the common limit range (CLR) property with respect to the mapping j. Moreover, if (f, j) is weakly compatible, then the functions f and j share a unique common fixed point in X.

Corollary 2.3. *Consider an neutrosophic metric space $(X, \mathscr{G}, \mathscr{D}, \mathscr{P}, \lhd, \rhd)$, where the t-norm is defined by $a \lhd b = \min\{a, b\}$ and the t-conorm by $a \oplus b = \max\{a, b\}$. Let f, g, h, and j be self-maps on X, and assume that for all $x, y \in X$ and for any $t > 0$,*

$$\int_0^{\mathscr{G}(fx,gy,s)} \hat{\mu}(s)ds \geq \mho \left(\int_0^{\eta(x,y,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{D}(fx,gy,s)} \hat{\mu}(s)ds \leq \phi \left(\int_0^{\zeta(x,y,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{P}(fx,gy,s)} \hat{\mu}(s)ds \leq \Phi \left(\int_0^{\epsilon(x,y,s)} \hat{\mu}(s)ds \right)$$

where $\hat{\mu} : \mathbb{R}^+ \to \mathbb{R}$ is a non-negative, summable function that is Lebesgue integrable. The set $\mho$ consists of functions $\mho : \mathbb{R} \to \mathbb{R}$ that are upper semi-continuous, satisfy $\mho(0) = 0$, and for which $\mho(s) < s$ holds for all $s > 0$.
Similarly, ϕ and Φ denote sets of comparison functions $\phi, \Phi : \mathbb{R} \to \mathbb{R}$ that are left-continuous, non-decreasing, and satisfy $\phi(s) > s$, and $\Phi(s) > s$ for every $s > 0$.
$\eta(x, y, s) = \mathscr{G}(fx, gy, s) \lhd \mathscr{G}(fx, hx, s) \lhd \mathscr{G}(gy, jy, s), \lhd\mathscr{G}(hx, gy, s), \lhd\mathscr{G}(fx, jy, s)$ and $\zeta(x, y, s) = \mathscr{D}(fx, gy, s) \oplus \mathscr{D}(fx, hx, s) \oplus \mathscr{D}(gy, jy, s), \oplus\mathscr{D}(hx, gy, s), \oplus\mathscr{D}(fx, jy, s)$ and $\epsilon(x, y, s) = \mathscr{P}(fx, gy, s) \oplus \mathscr{P}(fx, hx, s) \oplus \mathscr{P}(gy, jy, s), \oplus\mathscr{P}(hx, gy, s), \oplus\mathscr{P}(fx, jy, s)$.
Assume further that the pairs (f, h) and (g, j) satisfy the common limit range (CLR) property relative to the mappings h and j, respectively. Moreover, if both (f, h) and (g, j) are weakly compatible, then the mappings f, g, h, and j admit a unique common fixed point in X.

Corollary 2.4. *Let $(X, \mathscr{G}, \mathscr{D}, \mathscr{P}, \lhd, \rhd)$ be an neutrosophic metric space, where the t-norm is defined as $a \lhd b = \min\{a, b\}$ and the t-conorm as $a \oplus b = \max\{a, b\}$. Consider four self-mappings f, g, h, and j on X, and suppose that for all $x, y \in X$ and for any $t > 0$,*

$$\int_0^{\mathscr{G}(fx,gy,s)} \hat{\mu}(s)ds \geq \mho \left(\int_0^{\bar{\lambda}(x,y,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{D}(fx,gy,s)} \hat{\mu}(s)ds \leq \phi \left(\int_0^{\hat{\nu}(x,y,s)} \hat{\mu}(s)ds \right)$$

and

$$\int_0^{\mathscr{P}(fx,gy,s)} \hat{\mu}(s)ds \leq \Phi \left(\int_0^{W(x,y,s)} \hat{\mu}(s)ds \right)$$

where $\hat{\mu} : \mathbb{R}^+ \to \mathbb{R}$ is a Lebesgue integrable function that is summable and non-negative. The set $\mho$ consists of functions $\mho : \mathbb{R} \to \mathbb{R}$ which are upper semi-continuous, satisfy $\mho(0) = 0$, and for every $s > 0$, satisfy $\mho(s) < s$.
Similarly, ϕ and Φ refer to families of comparison functions $\phi, \Phi : \mathbb{R} \to \mathbb{R}$ that are left-continuous, non-decreasing, and fulfill $\phi(s) > s$ and $\Phi(s) > s$ for all $s > 0$.
$\bar{\lambda}(x, y, s) = \mathscr{G}(fx, gy, s) \lhd \mathscr{G}(fx, hx, s) \lhd \mathscr{G}(hx, jy, s), \lhd\mathscr{G}(gy, jy, s), \lhd\mathscr{G}(hx, gy, s), \lhd\mathscr{G}(fx, jy, s)$ and $\hat{\nu}(x, y, s) = \mathscr{D}(fx, gy, s)\oplus\mathscr{D}(fx, hx, s)\oplus\mathscr{D}(hx, jy, s), \oplus\mathscr{D}(gy, jy, s), \oplus\mathscr{D}(hx, gy, s), \oplus\mathscr{D}(fx, jy, s)$ and $W(x, y, s) = \mathscr{P}(fx, gy, s)\oplus\mathscr{P}(fx, hx, s)\oplus\mathscr{P}(hx, jy, s), \oplus\mathscr{P}(gy, jy, s), \oplus\mathscr{P}(hx, gy, s), \oplus\mathscr{P}(fx, jy, s)$.
Assume further that the pairs (f, h) and (g, j) satisfy the $(E.A)$ property, and that either $j(X)$ or $h(X)$ forms a closed subset of X. Moreover, if both pairs (f, h) and (g, j) are weakly compatible, then the mappings f, g, h, and j have a unique common fixed point in X.

3. Application to solvability of system of functional equations occuring in dynamical programming

In this section, we investigate the existence and uniqueness of solutions for a class of functional equations arising in dynamic programming, utilizing Theorem 2.1.

Let A and B be Banach spaces, with $C \subseteq A$ representing the state space and $D \subseteq B$ the decision space.

$$\left\{\begin{array}{l} S_1 s_{1(x)} = \text{ opt }_{y\in D}\left\{a(x,y) + A_1\left(x,y,s_1\left(\gamma_1(x,y)\right)\right)\right\} \\ S_2 s_{1(x)} = \text{ opt }_{y\in D}\left\{a(x,y) + B_1\left(x,y,s_1\left(\gamma_2(x,y)\right)\right)\right\} \\ S_3 s_{1(x)} = \text{ opt }_{y\in D}\left\{b(x,y) + A_2\left(x,y,s_1\left(\gamma_3(x,y)\right)\right)\right\} \\ S_4 s_{1(x)} = \text{ opt }_{y\in D}\left\{b(x,y) + B_2\left(x,y,s_1\left(\gamma_4(x,y)\right)\right)\right\} \\ S_5 s_{1(x)} = \text{ opt }_{y\in D}\left\{b(x,y) + A_3\left(x,y,s_1\left(\gamma_5(x,y)\right)\right)\right\} \\ S_6 s_{1(x)} = \text{ opt }_{y\in D}\left\{b(x,y) + B_3\left(x,y,s_1\left(\gamma_6(x,y)\right)\right)\right\} \end{array}\right\} \tag{*}$$

where $\gamma_1 : C \times D \to C, a : C \times D \to \mathbb{R}, A_i, B_i : C \times D \times \mathbb{R} \to \mathbb{R}$ and $s_1 \in E(C)$.
Denote by $\mathrm{E}(C)$ the space of all real-valued continuous functions defined on C. This space is equipped with a complete metric defined as

$$d(p,q) = \sup_{x\in C} |p(x) - \mathscr{P}(x)|$$

for every $p, q \in E(C)$.

As an application of Theorem 2.1, we investigate the existence of solutions to the following system of functional equations arising in dynamic programming.

$$\left\{\begin{array}{l} S_1(x) = \text{ opt }_{y\in D}\left\{a(x,y) + A_1\left(x,y,S_1\left(\gamma_1(x,y)\right)\right)\right\} \\ S_2(x) = \text{ opt }_{y\in D}\left\{a(x,y) + A_2\left(x,y,S_2\left(\gamma_2(x,y)\right)\right)\right\} \\ S_3(x) = \text{ opt }_{y\in D}\left\{b(x,y) + A_3\left(x,y,S_3\left(\gamma_3(x,y)\right)\right)\right\} \\ S_4(x) = \text{ opt }_{y\in D}\left\{b(x,y) + A_4\left(x,y,S_4\left(\gamma_4(x,y)\right)\right)\right\} \\ S_5(x) = \text{ opt }_{y\in D}\left\{b(x,y) + A_5\left(x,y,S_5\left(\gamma_5(x,y)\right)\right)\right\} \\ S_6(x) = \text{ opt }_{y\in D}\left\{b(x,y) + A_6\left(x,y,S_6\left(\gamma_6(x,y)\right)\right)\right\}. \end{array}\right\} \tag{**}$$

Theorem 3.1. *Let $A_1, A_2, A_3, B_1, B_2, B_3 : C \times D \times \mathbb{R} \to \mathbb{R}$ be bounded functions, and let $S_1, S_2, S_3, S_4, S_5, S_6 : E(C) \to E(C)$ be operators defined as in (*). Assume the following conditions are satisfied:*

(i) The pairs (S_1, S_4), (S_2, S_3), and (S_5, S_6) possess the $CLR_{S_4S_3}$ property.

(ii) If $S_1 s_1 = S_4 s_1$, then $S_1 S_4 s_1 = S_4 S_1 s_1$; similarly, if $S_2 s_1 = S_3 s_1$, then $S_2 S_3 s_1 = S_3 S_2 s_1$; and if $S_5 s_1 = S_6 s_1$, then $S_5 S_6 s_1 = S_6 S_5 s_1$.

(iii) The following inequalities hold:

$$\int_0^{|A_1(x,y,s_1(\gamma_1(x,y)))-B_1(x,y,s_2(\gamma_2(x,y)))|} \hat{\mu}(s)\,ds \geq \mho\left(\int_0^{\theta_1(s_1,s_2,s)} \hat{\mu}(s)\,ds\right),$$

$$\int_0^{|A_1(x,y,s_1(\gamma_1(x,y)))-B_1(x,y,s_2(\gamma_2(x,y)))|} \hat{\mu}(s)\,ds \leq \phi\left(\int_0^{\theta_2(s_1,s_2,s)} \hat{\mu}(s)\,ds\right),$$

and

$$\int_0^{|A_1(x,y,s_1(\gamma_1(x,y)))-B_1(x,y,s_2(\gamma_2(x,y)))|} \hat{\mu}(s)\,ds \leq \Phi\left(\int_0^{\theta_2(s_1,s_2,s)} \hat{\mu}(s)\,ds\right).$$

where

$$\theta_1(s_1,s_2,s) = \min\left\{\frac{s}{s+|S_1x-S_2y|}, \frac{s}{s+|S_1x-S_4x|}, \frac{s}{s+|S_4x-S_3y|}\right.$$
$$\left.\frac{s}{s+|S_2y-S_3y|}, \frac{s}{s+|S_4x-S_2y|}, \frac{s}{s+|S_1x-S_3y|}\right\}$$

and

$$\theta_2(s_1,s_2,s) = \max\left\{\frac{|S_1x-S_2y|}{s+|S_1x-S_2y|}, \frac{|S_1x-S_4x|}{s+|S_1x-S_4x|}, \frac{|S_4x-S_3y|}{s+|S_4x-S_3y|}\right.$$
$$\left.\frac{|S_2y-S_3y|}{s+|S_2y-S_3y|}, \frac{|S_4x-S_2y|}{s+|S_4x-S_2y|}, \frac{|S_1x-S_3y|}{s+|S_1x-S_3y|}\right\}$$

and

$$\theta_2(s_1,s_2,s) = \max\left\{\frac{|S_1x-S_2y|}{s+|S_1x-S_2y|}, \frac{|S_1x-S_4x|}{s+|S_1x-S_4x|}, \frac{|S_4x-S_3y|}{s+|S_4x-S_3y|}\right.$$
$$\left.\frac{|S_2y-S_3y|}{s+|S_2y-S_3y|}, \frac{|S_4x-S_2y|}{s+|S_4x-S_2y|}, \frac{|S_1x-S_3y|}{s+|S_1x-S_3y|}\right\}$$

Consequently, the system of functional equations possesses a unique common solution.

Proof. The system of equations admits a unique bounded solution if and only if the operators defined in (*) possess a unique common fixed point. Since the functions $A_1, A_2, A_3, B_1, B_2, B_3 : C\times D\times\mathbb{R}\to\mathbb{R}$ are bounded, there exists a constant $P \geq 0$ such that

$$\sup\{|A_i(x,y,z)|, |B_i(x,y,z)|; (x,y,z)\in C\times D\times\mathbb{R}\} \leq P.$$

Let $\in > 0, x \in C, \gamma_1, \gamma_2 \in E(c)$. Assume that opt $_{y\in D} = \sup_{y\in D}$. Then there exist $y_1, y_2 \in D$ such that

$$S_1s_1(x) < a(x,y_1) + A_1(x,y_1,s_1(\gamma_1(x,y_1))) + \epsilon \tag{11}$$
$$S_2s_2(x) < a(x,y_2) + B_1(x,y_2,s_2(\gamma_2(x,y_2))) + \epsilon \tag{12}$$

$$S_1s_1(x) \geq a(x,y_2) + A_1(x,y_2,s_1(\gamma_1(x,y_1))) \tag{13}$$
$$S_2s_2(x) \geq a(x,y_1) + B_1(x,y_1,s_2(\gamma_2(x,y_1))) \tag{14}$$

Equations (11) and (14) yields that

$$S_1s_1(x) - S_2s_2(x) < A_1(x,y_1,s_1(\gamma_1(x,y_1))) - B_1(x,y_1,s_2(\gamma_2(x,y_1))) + \epsilon$$

Similarly (12) and (13) yields that

$$S_1s_1(x) - S_2s_2(x) < A_1(x,y_2,s_1(\gamma_1(x,y_2))) - B_1(x,y_2,s_2(\gamma_2(x,y_2))) + \epsilon$$

Therefore,

$$S_1s_1(x) - S_2s_2(x) \leq \max\{|A_1(x,y_1,s_1(\gamma_1(x,y_1))) - B_1(x,y_1,s_2(\gamma_2(x,y_1)))|$$
$$|A_1(x,y_2,s_1(\gamma_1(x,y_2))) - B_1(x,y_2,s_2(\gamma_2(x,y_2)))|\} + \epsilon$$

Let $P = A_1(x,y_1,s_1(\gamma_1(x,y_1))), Q = B_1(x,y_1,s_2(\gamma_2(x,y_1)))$,
$R = A_1(x,y_2,s_1(\gamma_1(x,y_2)))$ and $S = B_1(x,y_2,s_2(\gamma_2(x,y_2)))$.
At this point, the previous inequality simplifies to

$$S_1s_1(x) - S_2s_2(x) \leq \max\{|P-Q|+\epsilon, |R-S|+\epsilon\} = H(\text{ say }) \tag{15}$$

Applying (15), we get

$$\begin{aligned}
&\int_0^{S_1s_1(x)-S_2s_2(x)} \hat{\mu}(s)ds \le \int_0^{H} \hat{\mu}(s)ds \\
&= \max\left\{\int_0^{|P-Q|+\epsilon} \hat{\mu}(s)ds, \int_0^{|R-S|+\epsilon} \hat{\mu}(s)ds\right\} \\
&= \max\left\{\int_0^{|P-Q|} \hat{\mu}(s)ds + \int_{|P-Q|}^{|P-Q|+\epsilon} \hat{\mu}(s)ds, \int_0^{|R-S|} \hat{\mu}(s)ds \int_{|R-S|}^{|R-S|+\epsilon} \hat{\mu}(s)ds\right\} \\
&= \max\left\{\int_0^{|P-Q|} \hat{\mu}(s)ds, \int_0^{|R-S|} \hat{\mu}(s)ds\right\} + \max\left\{\int_{|P-Q|}^{|P-Q|+\epsilon} \hat{\mu}(s)ds, \int_{|R-S|}^{|R-S|+\epsilon} \hat{\mu}(s)ds\right\}
\end{aligned}$$

Applying the third condition from Theorem 3.1, we derive

$$\int_0^{\frac{|S_1x-S_2y|}{s+|S_1x-S_2y|}} \hat{\mu}(s)ds \le \phi\left(\int_0^{\theta_2(s_1,s_2,s)} \hat{\mu}(s)ds\right) + \max\left\{\int_{|P-Q|}^{|P-Q|+\epsilon} \hat{\mu}(s)ds, \int_{|R-S|}^{|R-S|+\epsilon} \hat{\mu}(s)ds\right\}$$

For any $\epsilon > 0$, we can select $\mathscr{G}(L) \le \epsilon$ so that

$$\int \hat{\mu}(s)\, ds \le \delta,$$

where $\mathscr{G}(L)$ denotes the Lebesgue measure.

Now, the above inequality reduces to

$$\int_0^{\frac{|S_1x-S_2y|}{s+|S_1x-S_2y|}} \hat{\mu}(s)ds \le \phi\left(\int_0^{\theta_2(s_1,s_2,s)} \hat{\mu}(s)ds\right) + \delta$$

Taking the limit as $\delta \to 0$, we get

$$\int_0^{\mathscr{D}(S_1s_1,S_2s_2,s)} \hat{\mu}(s)ds \le \phi\left(\int_0^{\theta_2(s_1,s_2,s)} \hat{\mu}(s)ds\right)$$

and

$$\int_0^{\mathscr{P}(S_1s_1,S_2s_2,s)} \hat{\mu}(s)ds \le \Phi\left(\int_0^{\theta_2(s_1,s_2,s)} \hat{\mu}(s)ds\right)$$

Similarly, by applying the third condition of Theorem 3.1, it can be demonstrated that

$$\int_0^{\mathscr{G}(S_1s_1,S_2s_2,s)} \hat{\mu}(s)ds \ge \mho\left(\int_0^{\theta_2(s_1,s_2,s)} \hat{\mu}(s)ds\right)$$

Therefore, all the hypotheses of Theorem 3.1 are satisfied. Consequently, the system of functional equations $(**)$ admits a unique common solution. □

4. Application to Functional Equations in Dynamic Programming

In this section, we apply the main common fixed point theorem to establish the existence and uniqueness of solutions for a class of functional equations arising in dynamic programming.

Let S and D be nonempty sets representing the state space and decision space, respectively. Denote by $\mathcal{B}(S)$ the set of all bounded real-valued functions on S.

Consider the functional equation

$$u(x) = \sup_{y\in D}\left\{g(x,y) + H\big(x,y,u(\tau(x,y))\big)\right\}, \qquad x \in S, \tag{16}$$

where

(1) $g : S \times D \to \mathbb{R}$ is a bounded function,
(2) $\tau : S \times D \to S$ is a transition mapping,
(3) $H : S \times D \times \mathbb{R} \to \mathbb{R}$ is continuous.

Define the operator $T : \mathcal{B}(S) \to \mathcal{B}(S)$ by

$$(Tu)(x) = \sup_{y \in D} \left\{ g(x,y) + H\big(x, y, u(\tau(x,y))\big) \right\}, \qquad x \in S.$$

Then a function $u \in \mathcal{B}(S)$ is a solution of (16) if and only if it is a fixed point of T.

Now define a neutrosophic metric on $\mathcal{B}(S)$ by

$$M(u,v,t) = \frac{t}{t + \|u - v\|_\infty},$$

$$I(u,v,t) = \frac{\|u - v\|_\infty}{t + \|u - v\|_\infty},$$

and

$$F(u,v,t) = \frac{\|u - v\|_\infty}{t + \|u - v\|_\infty}, \qquad t > 0,$$

where

$$\|u - v\|_\infty = \sup_{x \in S} |u(x) - v(x)|.$$

Then,

$$(\mathcal{B}(S), M, I, F, *, \diamond)$$

is a complete neutrosophic metric space.

Assume that there exists a constant $k \in (0,1)$ such that

$$|H(x,y,r) - H(x,y,s)| \le k|r - s| \tag{17}$$

for all $x \in S$, $y \in D$, and $r, s \in \mathbb{R}$.

Further, define the mappings

$$A = T, \qquad B = T^2.$$

Theorem 4.1. *Suppose that condition (17) holds and that the mappings A and B satisfy all the assumptions of the main common fixed point theorem established in this paper. Then the functional equation (16) possesses a unique bounded solution in $\mathcal{B}(S)$.*

Proof. For any $u, v \in \mathcal{B}(S)$ and $x \in S$, we have

$$\begin{aligned}
|(Tu)(x) - (Tv)(x)| &= \Bigg| \sup_{y \in D} \Big\{ g(x,y) + H(x,y,u(\tau(x,y))) \Big\} \\
&\qquad - \sup_{y \in D} \Big\{ g(x,y) + H(x,y,v(\tau(x,y))) \Big\} \Bigg| \\
&\le \sup_{y \in D} |H(x,y,u(\tau(x,y))) - H(x,y,v(\tau(x,y)))| \\
&\le k \sup_{y \in D} |u(\tau(x,y)) - v(\tau(x,y))| \\
&\le k \|u - v\|_\infty.
\end{aligned}$$

Consequently,

$$\|Tu - Tv\|_\infty \le k \|u - v\|_\infty.$$

Hence,

$$M(Tu, Tv, t) = \frac{t}{t + \|Tu - Tv\|_\infty} \ge \frac{t}{t + k\|u - v\|_\infty} \ge M(u,v,t),$$

while

$$I(Tu, Tv, t) \leq I(u, v, t),$$

and

$$F(Tu, Tv, t) \leq F(u, v, t).$$

Thus, T satisfies a neutrosophic contractive condition. By the hypotheses, the mappings $A = T$ and $B = T^2$ satisfy the assumptions of the main common fixed point theorem. Therefore, A and B possess a unique common fixed point $u^* \in \mathcal{B}(S)$ such that

$$Au^* = Bu^* = u^*.$$

Since $A = T$, it follows that

$$Tu^* = u^*.$$

Hence, u^* is a solution of (16). The uniqueness of the common fixed point guarantees that this solution is unique. □

5. Conclusion

In this paper, we have established new common fixed point results for two pairs of weakly compatible mappings in neutrosophic metric spaces under an integral type contraction condition via the CLR property. These findings extend and unify several existing results in the fixed point theory literature. Moreover, we demonstrated the practical significance of our theoretical results by applying the common fixed point theorem to guarantee the existence and uniqueness of solutions to functional equations arising in dynamic programming for multistage decision processes. An illustrative example was also provided to validate the main results. Future research may explore further generalizations and applications in more complex neutrosophic structures.

Future Work

Building on the results presented in this paper, several directions for future research are envisioned:

(1) **Extension to Other Neutrosophic Structures:** Investigate common fixed point theorems in more generalized neutrosophic structures, such as neutrosophic normed spaces or neutrosophic probabilistic metric spaces.
(2) **Relaxation of Conditions:** Explore fixed point results under weaker compatibility or contractive conditions to broaden the applicability of the theory.
(3) **Multivalued Mappings:** Extend the current framework to multivalued (set-valued) mappings in neutrosophic metric spaces, which have practical significance in optimization and control theory.
(4) **Applications to Differential and Integral Equations:** Apply the fixed point theorems to establish existence and uniqueness of solutions to more complex functional, differential, and integral equations modeled within neutrosophic contexts.
(5) **Algorithm Development:** Design iterative algorithms based on the fixed point results for practical computation of solutions in applied problems, including decision sciences and dynamic systems.
(6) **Interdisciplinary Applications:** Explore applications in other fields such as machine learning, data analysis under uncertainty, and neutrosophic modeling in engineering and social sciences.

References

[1] Yildiz, C., Sharma, S., & Kutukcu, S. (2010). On some theorems in intuitionistic fuzzy metric spaces. *Journal Name*, 22, 44–49.
[2] Atanassov, K. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets and Systems, 20*(1), 87–96.
[3] Park, J. H. (2004). Intuitionistic fuzzy metric spaces. *Chaos, Solitons & Fractals, 22*, 1039–1046.

[4] Alaca, C., Turkoglu, D., & Yildiz, C. (2006). Fixed points in intuitionistic fuzzy metric spaces. *Chaos, Solitons & Fractals, 29*, 1073–1078.

[5] Turkoglu, D., Alaca, C., Cho, Y. J., & Yildiz, C. (2006). Common fixed point theorems in intuitionistic fuzzy metric spaces. *Journal of Applied Mathematics and Computing, 22*, 411–424.

[6] Turkoglu, D., Alaca, C., & Yildiz, C. (2006). Compatible maps and compatible maps of types (α) and (β) in intuitionistic fuzzy metric spaces. *Demonstratio Mathematica, 39*, 671–684.

[7] Jungck, G., & Rhoades, B. E. (1998). Fixed point for set-valued functions without continuity. *Indian Journal of Pure and Applied Mathematics, 29*, 227–238.

[8] Khan, M., Swaleh, M., & Sessa, S. (1984). Fixed point theorems by altering distances between the points. *Bulletin of the Australian Mathematical Society, 30*, 1–9.

[9] Wairojjana, N., Dosenović, T., Rakić, D., Gopal, D., & Kumam, P. (2015). An altering distance function in fuzzy metric fixed point theorems. *Fixed Point Theory and Applications, 2015*(69), 1–19.

[10] Sintunavarat, W., & Kumam, P. (2011). Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces. *Journal of Applied Mathematics, 2011*, Article ID 509230, 14 pages.

[11] Mihet, D. (2009). Altering distances in probabilistic Menger spaces. *Nonlinear Analysis: Theory, Methods & Applications, 71*, 2734–2738.

[12] Shen, Y., Qiu, D., & Chen, W. (2012). Fixed point theorems in fuzzy metric spaces. *Applied Mathematics Letters, 25*, 138–141.

[13] Chauhan, S., Sintunavarat, W., & Kumam, P. (2012). Common fixed point theorems for weakly compatible mappings in fuzzy metric spaces using (JCLR) property. *Applied Mathematics, 3*(9), 976–982.

[14] Branciari, A. (2002). A fixed point theorem for mappings satisfying a general contractive condition of integral type. *International Journal of Mathematics and Mathematical Sciences, 29*(9), 531–536.

[15] Murthy, P., Kumar, P., & Tas, S. (2010). Common fixed points of self-maps satisfying an integral type contractive condition in fuzzy metric spaces. *Mathematical Communications, 15*, 521–537.

[16] Alsulami, H., Karapınar, E., Regan, E., & Shahi, P. (2014). Fixed points of generalized contractive mappings of integral type. *Fixed Point Theory and Applications, 2014*, Article ID 213.

[17] Liu, Z., Han, Y., Kang, S., & Ume, J. (2014). Common fixed point theorems for weakly compatible mappings satisfying contractive conditions of integral type. *Fixed Point Theory and Applications, 2014*, Article ID 132.

[18] Sarwar, M., Zada, M., & Erhan, I. (2015). Common fixed point theorems of integral type contraction on metric spaces and its applications to system of functional equations. *Fixed Point Theory and Applications, 2015*(217), 1–15.

[19] Jain, M., Gupta, N., & Kumar, S. (2019). Common fixed point results for w-compatible mappings along with (CLRST) property in fuzzy metric spaces. *TWMS Journal of Applied and Engineering Mathematics, 9*(4), 851–863.

[20] Shatanawi, W., Gupta, V., & Kanwar, A. (2020). New results on modified intuitionistic generalized fuzzy metric spaces by employing E.A. property and common E.A. property for coupled maps. *Journal of Intelligent & Fuzzy Systems, 38*(3), 1–8.

[21] Kirişci, M., & Şimşek, N. (2020), Neutrosophic metric spaces. *Mathematical Sciences*, 14(3), 241–248.

[22] Jenifer, P., Jeyaraman, M., & Mursaleen, M. (2025). Weighted statistical convergence in neutrosophic normed linear spaces. *Neutrosophic Sets and Systems, 79*(1).